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MINIMAL LATTICE-SUBSPACES

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ABSTRACT. In this paper the existence of minimal lattice-subspaces of a vector lattice E containing a subset B of E_+ is studied (a lattice-subspace of E is a subspace of E which is a vector lattice in the induced ordering). It is proved that if there exists a Lebesgue linear topology τ on E and E_+ is τ-closed (especially if E is a Banach lattice with order continuous norm), then minimal lattice-subspaces with τ-closed positive cone exist (Theorem 2.5).

In the sequel it is supposed that $B = \{x_1, x_2, \dots, x_n\}$ is a finite subset of $C_+(\Omega)$, where Ω is a compact, Hausdorff topological space, the functions x_i are linearly independent and the existence of finite-dimensional minimal lattice-subspaces is studied. To this end we define the function $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$ where $r(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big)$. If $R(\beta)$ is the range of β and K the convex hull of the closure of $R(\beta)$, it is proved:

(i) There exists an m-dimensional minimal lattice-subspace containing B if and only if K is a polytope of $\mathbb{R}^n$ with m vertices (Theorem 3.20).

(ii) The sublattice generated by B is an m-dimensional subspace if and only if the set $R(\beta)$ contains exactly m points (Theorem 3.7).

This study defines an algorithm which determines whether a finite-dimensional minimal lattice-subspace (sublattice) exists and also determines these subspaces.

1. Introduction

It is known that $C[0,1]$ is a universal Banach space in the sense that every separable Banach space is isometric to a closed subspace of $C[0,1]$. In [11] it is shown that each separable Banach lattice is order-isomorphic to a closed lattice-subspace of $C[0,1]$; therefore $C[0,1]$ is also a universal Banach lattice. Since the sublattices of $C[0,1]$ are not enough for this representation, the lattice-subspaces seems to be the right class of subspaces for studying Banach lattices.

The structure of lattice-subspaces has not been systematically studied. In [7] it is shown that a subspace X of a vector lattice is a lattice-subspace if and only if there exists a positive projection from the vector sublattice generated by X onto X. In [10] and [11] the existence of positive bases in lattice-subspaces is studied. A survey of lattice-subspaces and positive projections, as well as some new results, is proved in [1]. In [12] the finite-dimensional lattice-subspaces of $C(\Omega)$ are studied.

In the present paper the existence of minimal lattice-subspaces of a vector lattice E which contains a subset B of E_+ is studied. In the theory of Banach lattices (and

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in applications) we are interested in a lattice-subspace of E containing B which is as "close" as possible to the linear subspace $[B]$ generated by B.

Such a subspace is the sublattice $S(B)$ generated by B (note that $S(B)$ is the minimum sublattice containing B and also that $S(B) = [B]^\vee - [B]^\vee$ where $[B]^\vee$ is the set of finite supremum of the elements of $[B]$) but $S(B)$ is in general a "big" subspace which is "very far" from $[B]$. In Example 3.18 $[B]$ is 3-dimensional, $S(B)$ is dense in $C(\Omega)$ but a 4-dimensional lattice-subspace containing B exists. In Example 3.21 it is shown that a minimum lattice-subspace containing B does not always exist.

An important question is "how far" a minimal lattice-subspace is from $[B]$. Motivated by this question we study the existence of finite-dimensional minimal lattice-subspaces. Especially we suppose that $B = \{x_1, x_2, \dots, x_n\}$ is a subset of $C_+(\Omega)$, the vectors x_i are linearly independent and we study the existence of finite-dimensional minimal lattice-subspaces of $C(\Omega)$ containing B. In the framework of this problem we study also the question whether $S(B)$ is a finite-dimensional subspace.

To study this problem we define the function $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$ where $r(t) = (x_1(t), x_2(t), \dots, x_n(t))$. This function defines a curve in the simplex Δ_n of $\mathbb{R}^n_+$ which we call basic curve of the functions x_i and is very important for our study.

In Theorem 3.7 it is proved that $S(B)$ is finite-dimensional if and only if the range $R(\beta)$ of β is finite and a positive basis of $S(B)$ is also determined. Hence we can determine whether $S(B)$ is finite-dimensional because it is very easy to check if $R(\beta)$ is finite or not. By the property that $S(B) = [B]^\vee - [B]^\vee$ we cannot conclude whether $S(B)$ is finite-dimensional and also we cannot determine a positive basis of $S(B)$.

In Theorem 3.10 it is proved that if the convex hull K of the closure of $R(\beta)$ is a polytope with m vertices, then an m-dimensional minimal lattice-subspace Y exists and a positive basis of Y is given. The determination of the basis of Y is based on the determination of the vertices of K.

In general it is difficult to study whether K is a polytope or not and determine its vertices. In Corollary 3.15 it is proved that if K is a polytope, $\beta(t_0)$ a vertex of K and t_0 an interior point of a curve c of Ω, then the derivative at t_0 (whenever it exists) of the restriction of β on c is equal to zero. If for example $\Omega \subseteq \mathbb{R}^l$ and the function β is defined on the whole set Ω, then the partial derivatives of β at t_0 are equal to zero whenever t_0 is an interior point of Ω and the derivatives at t_0 of the restriction of β on the parametric curves of $\partial(\Omega)$ are equal to zero, if $t_0 \in \partial(\Omega)$. Hence t_0 can be obtained as a solution of a system of equations.

This property helps us to determine a set of possible vertices of K, i.e., a subset G of $\mathbb{R}^n$ which contains the vertices of K, whenever K is a polytope. After the determination of G it is easier to study if K is a polytope or not (see Algorithm 3.17 and Example 3.18).

An interesting remark on the structure of the lattice-subspaces is also that a minimal lattice-subspace containing B is not necessarily a subspace of $S(B)$, Example 3.21.

Recently lattice-subspaces have been employed in economics [2], [3].

Let E be a (partially) ordered vector space with positive cone E_+ and X a subspace of E. The cone $X \cap E_+$ will be called the *induced cone* of X, and the ordering defined in X by this cone the *induced ordering.* We will denote by X_+ the induced cone of X, i.e., $X_+ = X \cap E_+$. An *ordered subspace* of E is a subspace of E ordered by the induced cone. A *lattice-subspace* of E is an ordered subspace of E which is also a vector lattice (Riesz space).

Let X be a lattice-subspace of E. Then, for each $x, y \in X$ we will denote by $x \triangledown y$ (resp. $x \vartriangle y$) the supremum (resp. infimum) of $\{x, y\}$ in X. It is clear that

$$x \vee y \leq x \triangledown y \quad \text{and} \quad x \vartriangle y \leq x \wedge y$$

whenever $x \vee y$, $x \wedge y$ exist. If E is a vector lattice and $x \triangledown y = x \vee y$ for any $x, y \in X$ then X is a sublattice (Riesz subspace) of E. Let E be an ordered Banach space with positive cone E_+. A sequence $\{e_n\}$ is a *positive basis* of E if $\{e_n\}$ is a (Schauder) basis of E and $E_+ = \{x = \sum_{i=1}^{\infty} \lambda_i e_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. A positive basis $\{e_n\}$ of E is unique (in the sense of a positive multiple). The following result (see [1] or [12]) is very important for the study of finite-dimensional lattice-subspaces. It can be proved either elementary or as a partial result of the Choquet-Kentall Theorem.

Theorem 1.1. *A finite-dimensional ordered vector space E is a vector lattice if and only if E has a positive basis.*

For notation and terminology not defined here we refer to [4, 6, 9].

2. Minimal lattice-subspaces

Let E be a vector lattice and $B \subseteq E_+, B \neq \emptyset$. Let L be the set of lattice-subspaces of E, each of which contains B. If $X \in L$ and for any $Y \in L$ it holds:

$$Y \subseteq X \Rightarrow Y = X,$$

then we will say that X is a *minimal lattice-subspace* of E containing B.

If E is a vector lattice, then the sublattice generated by B is the minimum sublattice containing B.

As we will show later (Example 3.21) even if $E = \mathbb{R}^m$ a minimum lattice-subspace of E containing B does not always exist. So we state the following question:

Problem 2.1. *Does a minimal lattice subspace of E containing B exist?*

Let P be a cone of a linear space F (i.e., P is a convex subset of F, $\lambda x \in P$ for each $x \in P$ and $\lambda \in \mathbb{R}_+$ and $P \cap (-P) = \{0\}$). Suppose that $x, y \in P$. If there exists $z \in P$ with the properties: $z - x, z - y \in P$ and for each $w \in P, w - x, w - y \in P$ imply that $w - z \in P$, then we will say that z is the supremum of $\{x, y\}$ in P and we will denote

$$z = \sup_P\{x, y\}.$$

The infimum of $\{x, y\}$ in P is defined analogously. If for each $x, y \in P$, $z = \sup_P\{x, y\}$ exists, then $\inf_P\{x, y\}$ also exists.

If P is a cone of a linear space F and for each $x, y \in P$ the supremum of $\{x, y\}$ exists in P, then we will say that P is a lattice cone of F.

If $x = x_1 - x_2$ where $x_1, x_2 \in P$, then it is easy to show that $\sup\{x, 0\} = \sup_P\{x_1, x_2\} - x_2$ is the supremum of $\{x_1, x_2\}$ in $X = P - P$. Therefore the following result holds.

A cone P of a vector space F is a lattice-cone if and only if the subspace $X = P - P$, ordered by the cone P, is a vector lattice.

In the next results of this paragraph we will suppose that E is a vector lattice equipped with a linear topology τ with the properties:

(i) E_+ is τ-closed;
(ii) each increasing, order bounded net of E has a τ-convergent subnet (i.e., the topology τ is Lebesgue).

Property (i) implies also that τ is Hausdorff because if we suppose that $x \in E$, $x \neq 0$ and $0 \in x + V$ for each open symmetric neighborhood V of zero, then $0 \in -x + V$; therefore x and $-x$ belong to E_+ and hence $x = 0$, contradiction.

If the topology τ is order continuous (i.e., each decreasing net of E with infimum zero is τ-convergent to zero) and E is Dedekind complete, then τ satisfies (ii). If the order intervals of E are τ-compact, the statement (ii) is also satisfied (for related results see [4, Theorem 11.13]). Hence, the weak star topology of a dual Banach lattice and the weak topology of a Banach lattice with order continuous norm [4, Theorem 12.9], have property (ii).

Proposition 2.2. *Let $(P_i)_{i\in I}$ be a decreasing net of τ-closed lattice cones of E_+ (i.e., $P_i \subseteq E_+$ and $i \preceq j \Rightarrow P_i \supseteq P_j$). Then $P = \bigcap_{i\in I} P_i$ is a τ-closed lattice cone of E.*

Proof. P is a τ-closed cone of E_+. Let $x, y \in P$. Denote by z_i the supremum of $\{x, y\}$ in P_i. For each $i, j \in I$ with $i \preceq j$ we have $P_j \subseteq P_i \subseteq E_+$; therefore,

$$x, y \leq z_i \leq z_j \leq x + y.$$

Since τ has property (ii), there exists a τ-convergent subnet of $(z_i)_{i\in I}$ which we will still denote by $(z_i)_{i\in I}$. This net is also increasing, and let $z = \lim_{i\in I} z_i$. Let $i \in I$. Then for each $j \in I$ with $i \preceq j$, we have:

$$z_j, z_j - x, z_j - y \in P_j \subseteq P_i.$$

Since the cone P_i is τ-closed, we have that

$$z, z - x, z - y \in P_i, \quad \text{for each } i \in I.$$

Therefore

$$z, z - x, z - y \in P.$$

Suppose that $w \in P$ with $w - x, w - y \in P$. Since $P \subseteq P_j$ we have that $w - z_j \in P_j \subseteq P_i$ for each $j \in I$ with $i \preceq j$. Hence $w - z \in P_i$ for each i; therefore $w - z \in P$. So we have proved that $z = \sup_P\{x, y\}$; therefore P is a lattice cone. □

Theorem 2.3. *Let $P \subseteq E_+$ be a cone and let $\Phi(P)$ be the set of τ-closed lattice cones of E_+ each of which contains P. Then $\Phi(P)$ has minimal elements.*

Proof. $\Phi(P) \neq \emptyset$ because $E_+ \in \Phi(P)$ and $\Phi(P)$, ordered by the relation "$\supseteq$", is a partially ordered set. Suppose that $\mathcal{F}$ is a totally ordered subset of $\Phi(P)$. Then by the previous result $Q = \bigcap_{A\in\mathcal{F}} A$ is a τ-closed lattice cone of E. By Zorn's Lemma the theorem is true. □

Proposition 2.4. *Let $(X_i)_{i\in I}$ be a decreasing net of lattice-subspaces of E with τ-closed positive cones. Let $X = \bigcap_{i\in I} X_i$, $Y = X_+ - X_+$ and $Y_+ = Y \cap E_+$. Then*

(i) *$X_+ = \bigcap_{i\in I} X_i^+$.*
(ii) *$Y \subseteq X$, $Y_+ = X_+$ and Y is a lattice-subspace of E with τ-closed positive cone.*

Proof. (i) $X_+ = X \cap E_+ = \big(\bigcap_{i\in I} X_i\big) \cap E_+ = \bigcap_{i\in I} X_i^+$.

(ii) $Y = X_+ - X_+ \subseteq X$. $Y_+ \subseteq X \cap E_+ = X_+$. Also $X_+ = X_+ - \{0\} \subseteq Y$; therefore $X_+ \subseteq Y_+$. Hence $X_+ = Y_+$. The net $(X_i^+)_{i\in I}$ is a decreasing net of τ-closed lattice cones of E_+; therefore Y_+ is a τ-closed lattice cone. Hence Y, is a lattice-subspace of E. □

Theorem 2.5. *Let $B \subseteq E_+$ and*

$$l(B) = \{Y \subseteq E \mid Y \text{ is a lattice-subspace, } Y_+ \text{ is } \tau\text{-closed and } B \subseteq Y\}.$$

Then $l(B)$ has minimal elements.

Proof. The set $l(B)$ is nonempty because it contains E. The set $l(B)$, ordered by the relation "$\supseteq$", is a partially ordered set. Let $\mathcal{F}$ be a totally ordered subset of $l(B)$. By the previous proposition there exists $Y \in l(B)$ such that $Y \subseteq A$ for each $A \in \mathcal{F}$. Therefore, by Zorn's Lemma $l(B)$ has minimal elements. □

Corollary 2.6. *Let E be a Banach lattice with order continuous norm and $B \subseteq E_+$. Then the set of lattice-subspaces of E with (norm) closed positive cone which contains B has minimal elements.*

3. The finite-dimensional case in $C(\Omega)$

In this paper we shall denote by Ω a compact, Hausdorff topological space and by $C(\Omega)$ the Banach lattice of continuous real valued functions defined on Ω.

We will also denote by $x_1, \dots, x_n$, n fixed linearly independent positive elements of $C(\Omega)$ and by X the subspace of $C(\Omega)$ generated by $x_1, \dots, x_n$, i.e.,

$$X = [x_1, x_2, \dots, x_n].$$

In [12] necessary and sufficient conditions in order for X to be a lattice-subspace of $C(\Omega)$ are given.

In this paper we study the problem:

Problem 3.1. *Does a finite-dimensional lattice-subspace (sublattice) of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$ exist?*

For each $x \in \mathbb{R}^m$ we will denote by $x(i)$ the i-coordinate of x, by $\|x\|_1$ the norm $\|x\|_1 = \sum_{i=1}^m |x(i)|$, by $\{e_1, e_2, \dots, e_m\}$ the usual basis of $\mathbb{R}^m$ and by Δ_m the simplex (base) of $\mathbb{R}^m_+$, i.e.,

$$\Delta_m = \{x \in \mathbb{R}^m_+ \mid \|x\|_1 = 1\}.$$

Also if $x \in \mathbb{R}^m, y \in \mathbb{R}^l$ we shall denote by (x, y) the vector z of $\mathbb{R}^{m+l}$ with $z(i) = x(i)$ for $i = 1, 2, \dots, m$ and $z(m+i) = b(i)$ for $i = 1, 2, \dots, l$. If A is an $m \times m$ matrix we shall denote by A^T the transpose and by A^{-1} the inverse matrix of A.

Let $y_1, y_2, \dots, y_m \in C_+(\Omega)$. Then we will call the function $v(t) = (y_1(t), y_2(t), \dots, y_m(t)), t \in \Omega$, the *curve* and the function $\gamma(t) = \frac{v(t)}{\|v(t)\|_1}, t \in \Omega$, with $v(t) \neq 0$, the *basic curve* of $y_1, y_2, \dots, y_m$. We will denote by $D(\gamma)$ the domain and by $R(\gamma)$ the range of γ. It is clear that $D(\gamma)$ is an open subset of Ω and $R(\gamma) \subseteq \Delta_m$.

In this paper we will denote by r the curve and by β the basic curve of $x_1, x_2, \dots, x_n$, i.e.,

$$r(t) = \big(x_1(t), x_2(t), \dots, x_n(t)\big), \quad t \in \Omega \quad \text{and } \beta(t) = \frac{r(t)}{\|r(t)\|_1}.$$

As usual if K is a subset of a topological space F, we shall denote by $\mathrm{int}(K)$ the interior, by $\bar{K}$ the closure and by $\partial(K)$ the boundary of K. Also whenever F is a linear topological space we shall denote by $\mathrm{co}\, K$ the convex hull of K, by $\overline{\mathrm{co}} K$ the closure of $\mathrm{co}\, K$ and by $\mathrm{ep}(K)$ the set of extreme points of K.

Proposition 3.2 ([12, Proposition 2.2])**.** *Let Y be a lattice-subspace of $C(\Omega)$ with a positive basis $\{b_1, b_2, \dots, b_n\}$. Then Y is a sublattice of $C(\Omega)$ if and only if the sets $b_i^{-1}(0, +\infty) = \{t \in \Omega \mid b_i(t) > 0\}$, $i = 1, 2, \dots, n$, are pairwise disjoint.*

Theorem 3.3 ([12, Theorem 3.6])**.** *The statements (i) and (ii) are equivalent:*

(i) *X is a lattice-subspace of $C(\Omega)$.*
(ii) *There exist n linearly independent vectors $P_1, P_2, \dots, P_n$ of $\mathbb{R}^n$, belonging to the closure of the range of β such that for each $t \in D(\beta)$ the vector $\beta(t)$ is a convex combination of the vectors $P_1, P_2, \dots, P_n$.*

If the statement (ii) is true, A is the $n \times n$ matrix whose ith column is the vector P_i and $b_1, b_2, \dots, b_n$ are the functions defined by the formula

$$(b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T, \tag{1}$$

then $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X.

Lemma 3.4. *The functions $y_i \in C_+(\Omega), i = 1, 2, \dots, m$, are linearly independent if and only if the space generated by the range of the basic curve γ of y_i, $i = 1, 2, \dots, m$, is $\mathbb{R}^m$.*

Proof. Let W be the subspace of $\mathbb{R}^m$ generated by $R(\gamma)$. Then W is also generated by the range of the curve v of y_i, $i = 1, 2, \dots, m$. Let $\{u_i = v(t_i) \mid i = 1, 2, \dots, l\}$ be a basis of W. Then $l \leq m$.

Suppose that the functions y_i are linearly independent. Then

$$v(t) = \sum_{i=1}^{l} \xi_i(t) u_i, \quad \text{for each } t \in \Omega;$$

therefore

$$y_j(t) = \sum_{i=1}^{l} \xi_i(t) u_i(j), \quad j = 1, 2, \dots, m, \tag{2}$$

where $u_i(j)$ is the j-coordinate of u_i. For each t, the vector $\big(\xi_1(t), \xi_2(t), \dots, \xi_l(t)\big)$ is the unique solution of the system (2); therefore the functions ξ_i as linear combinations of the functions y_i belong to $C(\Omega)$. By (2) we have also that

$$y_i \in L = [\xi_1, \xi_2, \dots, \xi_l], \quad \text{for each } i;$$

therefore $m \leq \dim L \leq l$. Hence $m = l$ and $W = \mathbb{R}^m$.

To prove the converse, suppose that $l = m$ and

$$\sum_{i=1}^{m} \lambda_i y_i = 0.$$

Then

$$\sum_{i=1}^{m} \lambda_i y_i(t_j) = 0 \quad \text{for each } j = 1, 2, \dots, m.$$

Since the vectors $v(t_i)$, $i = 1, 2, \dots, m$, are linearly independent, the system has the unique solution $\lambda_i = 0$ for each i; therefore the functions y_i are linearly independent. □

Sublattices.

Theorem 3.5. *Let $R(\beta) = \{P_1, P_2, \dots, P_n\}$. (By the previous lemma the vectors P_i are linearly independent and by Theorem 3.3 X is a lattice-subspace.) Let $\{b_1, b_2, \dots, b_n\}$ be the positive basis of X defined by (1) and let $I_i = b_i^{-1}(0, +\infty)$, for each i.*

Then the following statements hold:

(i) *X is a sublattice of $C(\Omega)$.*
(ii) *$I_i = \beta^{-1}(P_i)$ for each i and $D(\beta) = \bigcup_{i=1}^n I_i$.*
(iii) *If y_i, $i = 1, 2, \dots, m$, are linearly independent elements of X_+ and γ is the basic curve of y_i, $i = 1, 2, \dots, m$, then there exists $\Phi \subseteq \{1, 2, \dots, n\}$ such that*
  (a) *$D(\gamma) = \bigcup_{i \in \Phi} I_i$,*
  (b) *the function γ is constant on I_i for each $i \in \Phi$,*
  (c) *$m \le l \le n$, where l is the cardinal number of $R(\gamma)$.*

Proof. Let $z = \sum_{i=1}^n x_i$ and $B_i = \beta^{-1}(P_i)$, $i = 1, 2, \dots, n$. Then the sets B_i are pairwise disjoint and $D(\beta) = \bigcup_{i=1}^n B_i$. By (1) we have that

$$\frac{1}{z(t)} \big(b_1(t), b_2(t), \dots, b_n(t)\big)^T = A^{-1} \big(\beta(t)\big)^T.$$

Since $A^{-1} \cdot A = I$, the dot-product of the j-row of A^{-1} and the vector P_i is equal to 1 if $i = j$ and 0 whenever $i \neq j$; therefore

$$A^{-1} \big(\beta(t)\big)^T = (e_i)^T \quad \text{for each } t \in B_i,$$

where $\{e_1, e_2, \dots, e_n\}$ is the usual basis of $\mathbb{R}^n$. Therefore

$$\frac{1}{z(t)} \big(b_1(t), b_2(t), \dots, b_n(t)\big) = e_i \quad \text{for each } t \in B_i.$$

Hence for each $t \in B_i$ it holds:

$$z(t) = b_i(t) > 0 \quad \text{and} \quad b_j(t) = 0 \quad \text{for each } j \neq i.$$

So

$$B_i \subseteq I_i \quad \text{and} \quad B_i \cap I_j = \emptyset \quad \text{for each } j \neq i.$$

Suppose that $t \in I_i \setminus B_i$. Since $D(\beta) = \bigcup_{k=1}^n B_k$, $t \in B_j$ for exactly one $j \neq i$. Hence $I_i \cap B_j \neq \emptyset$, contradiction. Hence $B_i = I_i$ for each i, and by Theorem 3.2, X is a sublattice. We have also shown the statement (ii).

The basic curve γ is

$$\gamma(t) = \frac{1}{y(t)} \big(y_1(t), y_2(t), \dots, y_m(t)\big)$$

where $y = \sum_{i=1}^m y_i$. Let

$$y_j = \sum_{i=1}^n \mu_{ji} b_i, \quad j = 1, 2, \dots, m.$$

Then $y = \sum_{i=1}^{n} \mu_i b_i$ where $\mu_i = \sum_{j=1}^{m} \mu_{ji}$ for each i. Let $\Phi = \{i \mid \mu_i > 0\}$. Then it is clear that

$$D(\gamma) = \bigcup_{i \in \Phi} I_i.$$

If $i \in \Phi$ and $t \in I_i$, then

$$\gamma(t) = \frac{1}{\mu_i} (\mu_{1i}, \mu_{2i}, \dots, \mu_{mi}) = Q_i;$$

hence γ is constant on I_i. Therefore

$$R(\gamma) = \{Q_i \mid i \in \Phi\}.$$

Since Φ is a subset of $\{1, 2, \dots, n\}$, we have that $l \leq n$ and by Lemma 3.4, $m \leq l$. □

Theorem 3.6. *The following statements are equivalent:*

(i) *X is a sublattice of $C(\Omega)$.*
(ii) *$R(\beta) = \{P_1, P_2, \dots, P_n\}$.*

Proof. Let X be a sublattice of $C(\Omega)$ and let $\{b_1, b_2, \dots, b_n\}$ be a positive basis of X. Let $x_j = \sum_{i=1}^{n} \lambda_{ji} b_i$. Then $z = \sum_{j=1}^{n} x_j = \sum_{i=1}^{n} \lambda_i b_i$ where $\lambda_i = \sum_{j=1}^{n} \lambda_{ji}$. Then the sets

$$I_i = b_i^{-1}(0, +\infty), \quad i = 1, 2, \dots, n,$$

are pairwise disjoint by Proposition 3.2. Hence for each $t \in I_k$ we have $x_i(t) = \lambda_{ik} b_k(t)$ and $x(t) = \lambda_k b_k(t)$, and therefore

$$\beta(t) = \frac{1}{\lambda_k} (\lambda_{1k}, \lambda_{2k}, \dots, \lambda_{nk}) = P_k.$$

Also $D(\beta) = \bigcup_{i=1}^{n} I_i$ because $t \in D(\beta)$ iff $z(t) > 0$ iff $b_i(t) > 0$ for at least one i. Hence

$$R(\beta) = \{P_1, P_2, \dots, P_n\};$$

therefore the theorem is true. □

Theorem 3.7. *Let Z be the sublattice of $C(\Omega)$ generated by $x_1, x_2, \dots, x_n$ and let $m \in \mathbb{N}$. Then the statements (i) and (ii) are equivalent:*

(i) *$\dim(Z) = m$.*
(ii) *$R(\beta) = \{P_1, P_2, \dots, P_m\}$.*

If the statement (ii) is true, then Z is constructed as follows:

(a) Enumerate $R(\beta)$ so that its n first vectors are linearly independent. (Such an enumeration exists by Lemma 3.4.) Denote again by P_i, $i = 1, 2, \dots, m$, the new enumeration and let $I_i = \beta^{-1}(P_i)$, $i = 1, 2, \dots, m$.
(b) Define the functions

$$x_{n+k}(t) = a_k(t) \, \|r(t)\|_1, \quad t \in \Omega, \quad k = 1, 2, \dots, m - n,$$

where a_k is the characteristic function of I_{n+k}.
(c) $Z = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$.

Proof. Suppose that (ii) is true and the assumptions (a), (b) are satisfied. We shall show that (c) is true. It is clear that $m \geq n$. The sets I_i are open subsets of $D(\beta)$ because the sets $\{P_i\}$ are open subsets of $R(\beta)$. Also $D(\beta) = \bigcup_{i=1}^m I_i$. Since $D(\beta)$ is an open subset of Ω, the sets I_i are open, nonempty subsets of Ω. Also $\partial(I_i) \cap I_j = \emptyset$. Hence $\partial(I_i) \subseteq \Omega \setminus D(\beta)$; therefore $\|r(t)\|_1 = 0$ for each $t \in \partial(I_i)$. This implies that the functions x_{n+k} are continuous; therefore $x_{n+k} \in C_+(\Omega)$ for each k.

Let v be the curve and γ the basic curve of $x_i,\ i = 1, 2, \dots, m$. Then by the definition of x_{n+k} we have that

$$v(t) = \big(r(t), 0\big) \quad \text{for each } t \in \bigcup_{i=1}^n I_i$$

and

$$v(t) = \big(r(t), \|r(t)\|_1\, e_{i-n}\big) \quad \text{if } t \in I_i, i > n.$$

Let $t \in I_i$. Then

$$\gamma(t) = \big(\beta(t), 0\big) = (P_i, 0) = Q_i, \quad \text{if } i \leq n$$

and

$$\gamma(t) = \frac{1}{2}\,\big(\beta(t), e_{i-n}\big) = \frac{1}{2}\,(P_i, e_{i-n}) = Q_i, \quad \text{for each } i = n+1, \dots, m.$$

Since $D(\gamma) = D(\beta) = \bigcup_{i=1}^m I_i$, we have that

$$R(\gamma) = \{Q_i \mid i = 1, 2, \dots, m\}.$$

The vectors $Q_i,\ i = 1, 2, \dots, m$, are linearly independent. Hence the functions $x_i,\ i = 1, 2, \dots, m$, are also linearly independent; therefore the subspace Y generated by $x_i,\ i = 1, 2, \dots, m$, is an m-dimensional sublattice of $C(\Omega)$ by the previous theorem. Therefore $Z \subseteq Y$. Since $x_i,\ i = 1, 2, \dots, n$, are linearly independent elements of Z_+ and the cardinal number of $R(\beta)$ is m, by the statement (iii) of Theorem 3.5 we have that $m \leq \dim Z$. Therefore $\dim Z = m$; hence $Z = Y$.

Suppose now that the statement (i) is true. Then $x_i,\ i = 1, 2, \dots, n$, are linearly independent elements of Z_+; therefore by Theorem 3.5, there exist a nonempty subset Φ of $\{1, 2, \dots, m\}$ and nonempty, pairwise disjoint open subsets $I_i,\ i \in \Phi$, of Ω such that $D(\beta) = \bigcup_{i\in\Phi} I_i$ and β is constant on each I_i. Hence $R(\beta) = \{P_1, P_2, \dots, P_l\}$ where l is the cardinal number of Φ. By the same theorem we have also that $n \leq l \leq m$. As we have proved before, we can construct an l-dimensional sublattice Y of Ω containing $x_1, x_2, \dots, x_n$; therefore $Z \subseteq Y$ and $m \leq l$. Hence $l = m$ and therefore the statement (ii) is true. □

Lattice-subspaces. A subset K of $\mathbb{R}^l$ is a *polytope* if K is the convex hull of a finite subset of $\mathbb{R}^l$. The extreme points of K are called vertices of K.

Theorem 3.8. *Let Y be an l-dimensional lattice-subspace of $C(\Omega)$ containing x_1, $x_2, \dots, x_n$. Suppose that $\{b_1, b_2, \dots, b_l\}$ is a positive basis of Y,*

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_i = \sum_{j=1}^n \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

$$\Phi = \{i \mid \sigma_i \neq 0\},$$

$$P_i = \frac{1}{\sigma_i}(\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

and K is the convex hull of $\overline{R(\beta)}$. Then

(i) *$P_i \in \overline{R(\beta)}$ for each $i \in \Phi$.*

(ii) *K is a polytope with vertices $P_{i1}, P_{i2}, \dots, P_{im}$ where $n \leq m \leq l$ and $i_\nu \in \Phi$ for each $\nu = 1, 2, \dots, m$.*

Proof. Let $x_{n+1}, \dots, x_l \in Y_+$ such that

$$Y = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_l].$$

Let

$$x_i = \sum_{j=1}^{l} \lambda_{ij} b_j, \quad i = 1, 2, \dots, l,$$

$$s_i = \sum_{j=1}^{l} \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

and $v(t) = \big(x_1(t), x_2(t), \dots, x_l(t)\big)$, $t \in \Omega$. Then $\|v(t)\|_1 = \sum_{i=1}^{l} s_i b_i$ and the function

$$\gamma(t) = \frac{v(t)}{\|v(t)\|_1}, \quad \|v(t)\|_1 \neq 0,$$

is the basic curve of $x_1, x_2, \dots, x_l$. By [12, Proposition 2.3], for each $i = 1, 2, \dots, l$ there exists a sequence $(\omega_{i\nu})$ of Ω such that

$$\lim_{\nu\to\infty} \frac{b_j(\omega_{i\nu})}{b_i(\omega_{i\nu})} = 0, \quad \text{for each } j \neq i.$$

Then

$$\lim_{\nu\to\infty} \frac{x_j(\omega_{i\nu})}{\|v(\omega_{i\nu})\|_1} = \lim_{\nu\to\infty} \left(\frac{\sum_{k=1}^{l} \lambda_{jk} \frac{b_k}{b_i}}{\sum_{k=1}^{l} s_k \frac{b_k}{b_i}} \right)(\omega_{i\nu}) = \frac{\lambda_{ji}}{s_i},$$

therefore

$$\lim_{\nu\to\infty} \gamma(\omega_{i\nu}) = \frac{1}{s_i}(\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{li}) = M_i. \tag{3}$$

Let A be the $l \times l$ matrix with columns the vectors M_i, $i = 1, 2, \dots, l$. Then using the expansion of x_i relative to the positive basis of Y we get

$$(x_1, x_2, \dots, x_l)^T = A(s_1 b_1, s_2 b_2, \dots, s_l b_l)^T. \tag{4}$$

Since $\{x_1, x_2, \dots, x_l\}$ is also a basis of Y, we have that $\operatorname{rank} A = l$; therefore the vectors M_i, $i = 1, 2, \dots, l$, are linearly independent. Let

$$\gamma(t) = \sum_{i=1}^{l} \xi_i(t) M_i \tag{5}$$

be the expansion of $\gamma(t)$ relative to the basis $\{M_1, M_2, \dots, M_l\}$ of $\mathbb{R}^l$. Then

$$\big(\gamma(t)\big)^T = A\big(\xi_1(t), \xi_2(t), \dots, \xi_l(t)\big)^T$$

and by (4) we get

$$\big(\xi_1(t),\xi_2(t),\dots,\xi_l(t)\big)=\frac{1}{\|v(t)\|_1}\big(s_1b_1(t),s_2b_2(t),\dots,s_lb_l(t)\big).$$

Hence $\xi_i(t)\in\mathbb{R}_+$ and $\sum_{i=1}^{l}\xi_i(t)=1$. Therefore $\gamma(t)$ is a convex combination of $M_1,M_2,\dots,M_l$. Therefore

$$R(\gamma)\subseteq \mathrm{co}\{M_1,M_2,\dots,M_l\}.$$

Let $P(x)=\big(x(1),x(2),\dots,x(n)\big)$, $x\in\mathbb{R}^l$, be the natural projection of $\mathbb{R}^l$ onto $\mathbb{R}^n$. Then

$$P\left(\frac{s_i}{\sigma_i}M_i\right)=P_i,\quad \text{for each } i\in\Phi. \tag{6}$$

If $i\notin\Phi$, then $P(M_i)=0$, because $\sigma_i=0$ and therefore $\lambda_{ki}=0$ for each $k=1,2,\dots,n$. Also

$$\beta(t)=\frac{\|v(t)\|_1}{\|r(t)\|_1}P(\gamma(t)),\quad \text{for each } t\in D(\beta)\subseteq D(\gamma);$$

therefore by (5) we get

$$\beta(t)=\sum_{i\in\Phi}\frac{\|v(t)\|_1}{\|r(t)\|_1}\xi_i(t)\frac{\sigma_i}{s_i}P_i.$$

Since $\beta(t)$ and P_i belong to the simplex Δ_n of $\mathbb{R}^n_+$, we have that $\beta(t)$ is a convex combination of the vectors $P_i,\ i\in\Phi$; hence

$$R(\beta)\subseteq \mathrm{co}\{P_i\mid i\in\Phi\}=L.$$

Since Φ is finite, the set L is closed; hence $\overline{R(\beta)}\subseteq L$. We shall show that $P_i\in\overline{R(\beta)}$, for each $i\in\Phi$. By (3) and (6) we have that $P\big(\frac{s_i}{\sigma_i}\gamma(\omega_{i\nu})\big)\to P_i$. Since $P_i\neq 0$, we have that $P(\gamma(\omega_{i\nu}))\neq 0$, for each ν. Therefore $r(\omega_{i\nu})=\|v(\omega_{i\nu})\|_1 P(\gamma(\omega_{i\nu}))\neq 0$; hence $\omega_{i\nu}\in D(\beta)$, for each ν. Similarly with the proof of (3) we can show that $P_i=\lim\beta(\omega_{i\nu})$. Hence $P_i\in\overline{R(\beta)}$; therefore $K=L$. Also $\mathrm{ep}(K)\subseteq\{P_i\mid i\in\Phi\}$. Hence

$$\mathrm{ep}(K)=\{P_{i1},P_{i2},\dots,P_{im}\}$$

where $i_\nu\in\Phi$ for $\nu=1,2,\dots,m$; therefore

$$K=\mathrm{co}\{P_{i1},P_{i2},\dots,P_{im}\}.$$

By Lemma 3.4, the subspace generated by $R(\beta)$, and therefore also by K, is the space $\mathbb{R}^n$. Hence $\mathrm{ep}(K)$ contains at least n vectors; therefore $n\le m\le l$. □

Theorem 3.9 ([5, Theorem 2]). *Let $d_1,d_2,\dots,d_m\in\mathbb{R}^l$ and let the polytope $D=\mathrm{co}\{d_1,d_2,\dots,d_m\}$. Then there exist non-negative, real-valued continuous functions $\xi_1,\xi_2,\dots,\xi_m$ defined on D such that $x=\sum_{i=1}^m\xi_i(x)d_i$ and $\sum_{i=1}^m\xi_i(x)=1$, for each $x\in D$.*

The previous result in a more general form is given also in [8].

Theorem 3.10. *Let the set $K=\mathrm{co}\,\overline{R(\beta)}$ be a polytope with vertices $P_1,P_2,\dots,P_m$. Suppose that the n first vertices $P_1,P_2,\dots,P_n$ of K are linearly independent*[1]*. Suppose also that $\xi_i,\ i=1,2,\dots,m$, are positive continuous real-valued functions defined on $D(\beta)$ such that $\sum_{i=1}^m\xi_i(t)=1$ and $\beta(t)=\sum_{i=1}^m\xi_i(t)P_i$, for each $t\in D(\beta)$.*

[1] A such enumeration of the vertices of K exists by Lemma 3.4.

Let $x_{n+i},\ i=1,2,\ldots,m-n$, *be the functions* $x_{n+i}(t)=\xi_{n+i}(t)\,\|r(t)\|_1$ *for each* $t\in D(\beta)$ *and* $x_{n+i}(t)=0$ *if* $t\notin D(\beta)$. *Then*

$$Y=[x_1,x_2,\ldots,x_n,x_{n+1},\ldots,x_m]$$

is a minimal lattice-subspace of $C(\Omega)$ *containing* $x_1,x_2,\ldots,x_n$ *and* $\dim Y=m$.

A positive basis $\{b_1,b_2,\ldots,b_m\}$ *of* Y *is given by the formula*

$$(b_1,b_2,\ldots,b_m)^T=A^{-1}\,(x_1,x_2,\ldots,x_m)^T$$

where A *is the* $m\times m$ *matrix with columns the vectors* $R_i, i=1,2,\ldots,m$, *defined below, in* (8).

Proof. We shall show that Y is a lattice-subspace of $C(\Omega)$. Let $v(t)=\big(x_1(t),x_2(t),\ldots,x_m(t)\big)$, $\gamma(t)=\frac{v(t)}{\|v(t)\|_1}$ and $l=m-n$. Then

$$\begin{aligned} v(t) &= \big(r(t),0\big)+\big(0,\sum_{i=1}^{l}\xi_{n+i}(t)\|r(t)\|_1e_i\big)\\ &= \|r(t)\|_1\sum_{i=1}^{m}\xi_i(t)\,(P_i,0)+\|r(t)\|_1\sum_{i=1}^{l}\xi_{n+i}(t)\,(0,e_i)\\ &= \|r(t)\|_1\sum_{i=1}^{m}\xi_i(t)\,M_i,\quad \text{for each } t\in D(\beta) \qquad (7)\end{aligned}$$

where M_i are the following vectors of $\mathbb{R}^m$:

$$M_i=(P_i,0)\quad\text{for } i=1,2,\ldots,n$$

and

$$M_i=(P_{n+i},e_i)\quad\text{for } i=1,2,\ldots,l.$$

The vectors M_i are linearly independent with $\|M_i\|_1=1$ for $i=1,2,\ldots,n$ and $\|M_i\|_1=2$ for $i=n+1,\ldots,m$. Hence $\|v(t)\|_1=\|r(t)\|_1\,g(t)$, where $g(t)=\sum_{i=1}^m\xi_i(t)\,\|M_i\|_1=1+\sum_{i=n+1}^m\xi_i(t)$. Therefore, by (7) we have

$$\gamma(t)=\frac{1}{g(t)}\sum_{i=1}^{m}\xi_i(t)\,\|M_i\|_1\,R_i,\quad\text{where } R_i=\frac{M_i}{\|M_i\|_1}. \qquad (8)$$

Hence $\gamma(t)$ is a convex combination of $R_i,\ i=1,2,\ldots,m$. We shall show that $R_i\in\overline{R(\gamma)}$ for each i. If $P_i=\beta(t_i)$, then $P_i=\sum_{j=1}^m\xi_j(t_i)P_j$ and by our assumption that P_i is an extreme point of K, we have that $\xi_i(t_i)=1$ and $\xi_j(t_i)=0$ for each $j\neq i$. Hence by (8) we have

$$\gamma(t_i)=\frac{1}{g(t_i)}\,\|M_i\|_1\,R_i=R_i.$$

If $P_i\notin R(\beta)$, then there exists a sequence (ω_ν) of $D(\beta)$ such that

$$P_i=\lim_{\nu\to\infty}\beta(\omega_\nu).$$

Then

$$\beta(\omega_\nu)=\sum_{j=1}^{m}\xi_j(\omega_\nu)\,P_j.$$

Since $0 \leq \xi_j(\omega_\nu) \leq 1$, there exists a subsequence of (ω_ν), which we will denote again by (ω_ν) such that

$$\lambda_j = \lim_{\nu\to\infty} \xi_j(\omega_\nu), \quad \text{for each } j = 1, 2, \dots, m.$$

Hence

$$P_i = \sum_{j=1}^{m} \lambda_j P_j,$$

which implies that $\lambda_i = 1$ and $\lambda_j = 0$ for each $j \neq i$, because P_i is an extreme point of K. By (8) and the definition of g we have that

$$\lim_{\nu\to\infty} \gamma(\omega_\nu) = R_i.$$

So by Theorem 3.3, Y is a lattice-subspace and a positive basis of Y is as in the formulation of the theorem.

Suppose that $Z \subseteq Y$ is a lattice-subspace containing $x_1, x_2, \dots, x_n$ and let $\dim Z = l$. Then $l \leq m$. By Theorem 3.8 the number m of vertices of K is less than or equal to l; therefore $m = l$. Hence $Z = Y$; therefore Y is minimal. □

Definition 3.11. *Let C be a convex subset of a normed space E. We shall say that x_0 is a conic point of C if x_0 is an extreme point of C, $C \setminus \{x_0\} \neq \emptyset$, and there exists a real number $\rho > 0$ such that*

$$x_0 + \rho \frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

Proposition 3.12. *Let D be a convex subset of a normed space E and $x_0 \in E$. If[2] $d = d(x_0, D) > 0$ and $C = \operatorname{co}(\{x_0\} \cup D)$, then x_0 is a conic point of C. (If D is bounded and closed, then C is also bounded and closed.)*

Proof. Let $x \in C, x \neq x_0$. Then $x = \lambda x_0 + (1-\lambda)y$, where $y \in D$ and $\lambda \in [0,1]$. Hence $x - x_0 = (1-\lambda)(y - x_0)$; therefore

$$\|x - x_0\| = (1-\lambda)\,\|y - x_0\| \geq (1-\lambda)\,d.$$

Also $x_0 + l(y - x_0) \in C$ for each $l \in [0,1]$. Therefore

$$x_0 + d\frac{x - x_0}{\|x - x_0\|} = x_0 + \frac{d(1-\lambda)}{\|x - x_0\|}\,(y - x_0) \in C.$$

To show that x_0 is an extreme point of C suppose that $x_0 = \frac{x_1+x_2}{2}$ where $x_1, x_2 \in C$ and $x_1, x_2 \neq x_0$. Then $x_i = \lambda_i x_0 + (1-\lambda_i)y_i$ with $\lambda_i \in (0,1)$ and $y_i \in D$. Then $x_0 = \frac{1}{2-\lambda_1-\lambda_2}\big((1-\lambda_1)y_1 + (1-\lambda_2)y_2\big) \in D$, contradiction. Hence x_0 is a conic point of C. □

Example 3.13. (i) For each cone $P \neq \{0\}$ of a normed space, 0 is a conic point of P.

(ii) Let C be a closed, convex, bounded subset of a Banach space E and let x_0 be an extreme point of C. If $C = \overline{\operatorname{co}}\,\operatorname{ep}(C)$ (i.e., C is the closure of the convex hull of the extreme points of C) and $x_0 \notin D = \overline{\operatorname{co}}\big(\operatorname{ep}(C) \setminus \{x_0\}\big)$, then $C = \operatorname{co}(\{x_0\} \cup D)$; therefore x_0 is a conic point of C.

(iii) Each vertex of a polytope C of $\mathbb{R}^m$ is a conic point of C.

[2]With $d(x_0, D)$ we denote the distance from x_0 to D.

We prove below that the tangent vector of a curve of C at a conic point of C is equal to zero.

Proposition 3.14. *Let C be a closed, convex subset of a normed space E and x_0 be a conic point of C. Let $\phi : (-\epsilon, \epsilon) \to C$ be a function with $\phi(0) = x_0$ where ϵ is a positive real number. Then*

$$\phi'(0) = 0,$$

whenever the derivative $\phi'(0)$ exists.

Proof. Let $\phi'(0) = \lim_{t\to 0} \frac{\phi(t)-\phi(0)}{t} \neq 0$. Then there exists $\delta > 0$ such that $\phi(t) \neq \phi(0)$ for each $|t| < \delta$. Hence

$$\lim_{t\to 0_+} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = \lim_{t\to 0_+} \frac{\phi(t) - \phi(0)}{t} \cdot \lim_{t\to 0_+} \frac{1}{\left\|\frac{\phi(t)-\phi(0)}{t}\right\|} = \frac{\phi'(0)}{\|\phi'(0)\|},$$

and similarly

$$\lim_{t\to 0_-} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = -\frac{\phi'(0)}{\|\phi'(0)\|}.$$

Since x_0 is a conic point of C, there exists $\rho > 0$ such that

$$x_0 + \rho\frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

Therefore

$$\lim_{\nu\to\infty} \left(\phi(0) + \rho\frac{\phi(1/\nu) - \phi(0)}{\|\phi(1/\nu) - \phi(0)\|}\right) = x_0 + \rho\frac{\phi'(0)}{\|\phi'(0)\|} = z_1 \in C$$

and

$$\lim_{\nu\to\infty} \left(\phi(0) + \rho\frac{\phi(-1/\nu) - \phi(0)}{\|\phi(-1/\nu) - \phi(0)\|}\right) = x_0 - \rho\frac{\phi'(0)}{\|\phi'(0)\|} = z_2 \in C.$$

Hence $x_0 = \frac{1}{2}(z_1 + z_2)$, contradiction. Therefore $\phi'(0) = 0$. □

Corollary 3.15. *Let the set $K = \operatorname{co}\overline{R(\beta)}$ be a polytope of $\mathbb{R}^n$ and let $\beta(t_0)$ be a vertex of K. If ϵ is a positive real number and $g : (-\epsilon, \epsilon) \to \Omega$ is a function with $g(0) = t_0$ and $\phi(\lambda) = \beta(g(\lambda))$, then*

$$\phi'(0) = 0,$$

whenever the derivative exists.

Remark 3.16. Suppose that there exists a finite-dimensional lattice-subspace of $C(\Omega)$ containing X. Then K is a polytope of $\mathbb{R}^n$. Suppose that $\beta(t_0)$ is a vertex of K. If c is a curve of Ω and t_0 an interior point of c, then the derivative at t_0 of the restriction of β on the curve c is equal to zero.

If for example $\Omega \subseteq \mathbb{R}^l$, then the partial derivatives of β at t_0 are equal to zero whenever $t_0 \in \operatorname{int}(\Omega)$. If $t_0 \in \partial(\Omega)$, the derivatives at t_0 of the restriction of β on the parametrics curves of $\partial(\Omega)$ are equal to zero.

Algorithm 3.17. *Theorem 3.10 and Corollary 3.15 define a process which in many cases, especially when $\Omega \subseteq \mathbb{R}^l$, determines whether a finite dimensional minimal lattice-subspace exists and determines also a positive basis of these subspaces. To study this problem we study if K is a polytope or not.*

If the set $R(\beta)$ is closed, then each extreme point (vertex) P_0 of $K = \operatorname{co} R(\beta)$ belongs to $R(\beta)$; therefore $P_0 = \beta(t_0)$. Also the geometry of the boundary of $D(\beta)$ and the differentiability of the functions x_i are very important for this study.

Let $\Omega = [a, b]$, the functions x_i are differentiable and $D(\beta) = \Omega$. Suppose that the set K is a polytope with vertices $\beta(t_i)$, $i = 1, 2, \dots, m$. Then at least $m - 2$ of t_i belong to (a, b); therefore the equation

$$\beta'(t) = 0, \tag{9}$$

where β' is the derivative of β, has at least $m-2$ roots in (a, b). Hence the vertices of K belong to the set

$$G = \{\beta(t) | t = a, t = b, \text{ or } t \text{ is a root of (9)}\}$$

which we call the set of possible vertices of K. Let $D = \operatorname{co} G$. It is easy to show that K is a polytope if and only if D is a polytope and $R(\beta) \subseteq D$.

Hence in this case the algorithm is the following:

(i) *Determine equation (9). If this equation does not have at least $n - 2$ roots in (a, b), then K is not a polytope.*
(ii) *Determine the roots t_i of (9) in (a, b).*
(iii) *We study whether $R(\beta) \subseteq D$. So we study whether $\beta(t)$ is a convex combination of $\beta(a), \beta(b), \beta(t_i)$, for each i. If $R(\beta) \nsubseteq D$, then K is not a polytope.*
(iv) *Determine the vertices of K and a positive basis of the minimal lattice-subspace, in accordance with Theorem 3.10.*

We give three examples below. In (i) it is shown that a finite-dimensional minimal lattice-subspace does not always exist. In (ii) we consider three elements x_1, x_2, x_3 of $C(\Omega)$, where Ω is a square of $\mathbb{R}^2$. We show that a 4-dimensional minimal lattice-subspace Y exists and a positive basis of Y is determined. We also remark that the sublattice generated by the elements x_i is dense in $C(\Omega)$. In (iii) the functions x_i are as in (ii), but Ω is a circle of $\mathbb{R}^2$. It is shown that a finite-dimensional minimal lattice-subspace does not exist. This difference between (ii) and (iii) depends on the geometry of the boundary of Ω.

Example 3.18. (i) Let $\Omega = [0, 1]$, $x_1(t) = 1, x_2(t) = t, x_3(t) = t^2$. Then

$$\beta(t) = \left(\frac{1}{1+t+t^2}, \frac{t}{1+t+t^2}, \frac{t^2}{1+t+t^2}\right), \quad t \in [0, 1],$$

is the basic curve of x_1, x_2, x_3 and $\beta'(t) \neq 0$ for each $t \in (0, 1)$. Suppose that Y is a finite-dimensional lattice-subspace of $C(\Omega)$ containing the functions x_i. Then $\dim Y \geq 3$, and therefore by Theorem 3.8 K is a polytope of $\mathbb{R}^3$ with at least three vertices, $\beta(t_1), \beta(t_2), \beta(t_3)$. Hence $\beta'(t) = 0$ for at least one point of $(0, 1)$, contradiction.

(ii) Let $\Omega = [0, 1] \times [0, 1]$, $x_1(u, v) = 1$, $x_2(u, v) = u$, $x_3(u, v) = v$ and $X = [x_1, x_2, x_3]$. Then

$$\beta(u, v) = \left(\frac{1}{1+u+v}, \frac{u}{1+u+v}, \frac{v}{1+u+v}\right), \quad (u, v) \in \Omega,$$

is the basic curve of x_1, x_2, x_3 and let $K = \operatorname{co} R(\beta)$. Since the range of β is not finite, the sublattice Z generated by x_1, x_2, x_3 is an infinite-dimensional subspace of $C(\Omega)$, Theorem 3.7. In this example we can also show that Z is dense in $C(\Omega)$ because Z is a sublattice of $C(\Omega)$ and Z contains the constant functions.

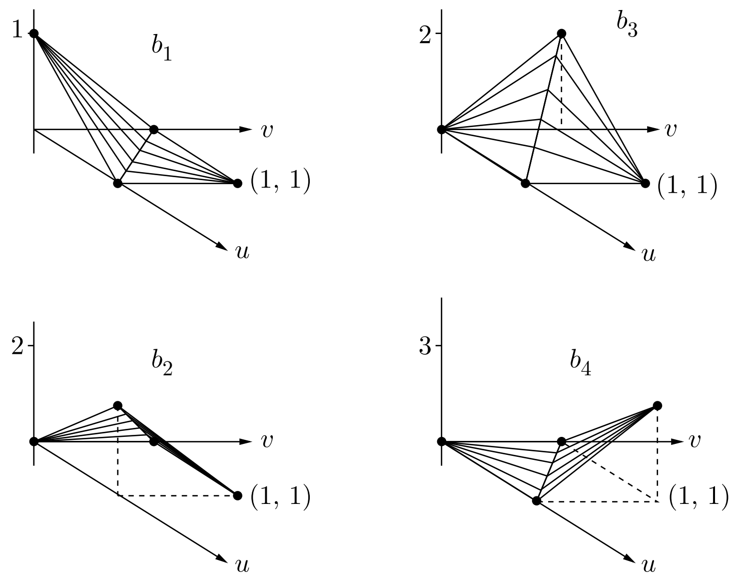

FIGURE 1

In order to study the existence of minimal lattice-subspaces we study whether the set K is a polytope of $\mathbb{R}^3$. To this end suppose that K is a polytope. Then by Theorem 3.8, K has at least three vertices and let $\beta(t_0)$ be a vertex of K. Then t_0 is also a vertex of Ω because in the contrary case t_0 will be an interior point of a line segment parallel to an axis of $\mathbb{R}^2$; therefore, and by the previous corollary, at least one of the partial derivatives of β at t_0 will be equal to zero, contradiction. Hence the points $P_1 = \beta(0,0) = (1,0,0)$, $P_2 = \beta(1,0) = (1/2,1/2,0)$, $P_3 = \beta(0,1) = (1/2,0,1/2)$ and $P_4 = \beta(1,1) = (1/3,1/3,1/3)$ define the set of possible vertices of K. Let $D = \mathrm{co}\{P_1,P_2,P_3,P_4\}$. From the above remarks we have that K is a polytope if and only if $K = D$ or equivalently if $R(\beta) \subseteq D$. It is easy to show that

$$\beta(u,v) = \sum_{i=1}^{4} \xi_i(u,v)\, P_i,$$

where $\xi_1 \in C(\Omega)$, $\xi_2(u,v) = 2\big(\frac{1-v}{1+u+v} - \xi_1(u,v)\big)$, $\xi_3(u,v) = 2\big(\frac{1-u}{1+u+v} - \xi_1(u,v)\big)$ and $\xi_4(u,v) = 3\big(\frac{u+v-1}{1+u+v} + \xi_1(u,v)\big)$.

Since $\beta(u,v)$ and the points P_i belong to the plane $x(1)+x(2)+x(3) = 1$ of $\mathbb{R}^3$ we have that $\sum_{i=1}^{4} \xi_i(u,v) = 1$. If $\xi(u,v) = \frac{1-u-v}{1+u+v}$ and if we put $\xi_1 = \xi^+$, then the functions ξ_i, $i = 1,2,3,4$, are positive and continuous; therefore $R(\beta) \subseteq D$. Hence K is a polytope with vertices P_i, $i = 1,2,3,4$, and the three first of them are linearly independent. By Theorem 3.10,

$$Y = [x_1, x_2, x_3, x_4],$$

where $x_4(u,v) = \xi_4(u,v)\,\|r(u,v)\|_1 = 3(1-u-v)^+$, is a minimal lattice-subspace containing x_1, x_2, x_3.

A positive basis $\{b_1,b_2,b_3,b_4\}$ of Y is given by the formula

$$(b_1,b_2,b_3,b_4)^T = A^{-1}\,(x_1,x_2,x_3,x_4)^T,$$

where A is the 4×4 matrix with columns the vectors $R_i = \frac{M_i}{\|M_i\|_1}$, $i = 1, 2, 3, 4$, and $M_1 = (P_1, 0) = (1, 0, 0, 0)$, $M_2 = (P_2, 0) = (1/2, 1/2, 0, 0)$, $M_3 = (P_3, 0) = (1/2, 0, 1/2, 0)$, $M_4 = (P_4, e_1) = (1/3, 1/3, 1/3, 1)$.

After the computations we get

$$\begin{aligned} b_1(u,v) &= x_1 - x_2 - x_3 + \frac{1}{3}x_4 = \begin{cases} 1-u-v & | \ u+v \le 1, \\ 0 & | \ u+v > 1, \end{cases} \\ b_2(u,v) &= 2x_2 - \frac{2}{3}x_4 = \begin{cases} 2u & | \ u+v \le 1, \\ 2(1-v) & | \ u+v > 1, \end{cases} \\ b_3(u,v) &= 2x_3 - \frac{2}{3}x_4 = \begin{cases} 2v & | \ u+v \le 1, \\ 2(1-u) & | \ u+v > 1, \end{cases} \\ b_4(u,v) &= 2x_4 = \begin{cases} 0 & | \ u+v \le 1, \\ 3(u+v-1) & | \ u+v > 1 \end{cases} \quad \text{(Figure 1).} \end{aligned}$$

(iii) Let $\Omega = \{(u,v) \in \mathbb{R}^2 | u^2 + v^2 \le 1\}$ and let x_i, $i = 1, 2, 3$, be the functions of the previous example. Suppose that K is a polytope and $\beta(t_0)$ a vertex of K. As before we have that $t_0 \in \partial(\Omega)$ and let $t_0 = (\cos\theta_0, \sin\theta_0)$. Then by the corollary we have $\phi'(\theta_0) = 0$ where $\phi(\theta) = \beta(\cos\theta, \sin\theta)$. This is a contradiction because $\phi'(\theta) \neq 0$ for each θ. Therefore a finite-dimensional lattice-subspace containing the functions x_i does not exist.

To study subspaces of $\mathbb{R}^l$, $l > 1$, suppose that $\Omega = \{1, 2, \dots, l\}$. Then $C(\Omega) = \mathbb{R}^l$,

$$x_i = \big(x_i(1), x_i(2), \dots, x_i(l)\big), \quad i = 1, 2, \dots, n,$$

are linearly independent, positive elements of $\mathbb{R}^l$ and

$$X = [x_1, x_2, \dots, x_n].$$

The curve r and the basic curve β of the vectors x_i, $i = 1, 2, \dots, n$, are the functions:

$$r(i) = \big(x_1(i), x_2(i), \dots, x_n(i)\big), \quad i = 1, 2, \dots, l,$$

and

$$\beta(i) = \frac{r(i)}{\|r(i)\|_1}, \quad \text{for each } i \text{ with } \|r(i)\|_1 \neq 0.$$

Let m be the cardinal number of $R(\beta)$. Then $m \le l$ and by Lemma 3.4, $n \le m$; therefore $n \le m \le l$. Let K be the convex hull of $R(\beta)$. Then K, as the convex hull of a finite subset of $\mathbb{R}^n$, is a polytope with d vertices. It is clear that

$$n \le d \le m \le l$$

and that each vertex of K belongs to $R(\beta)$. Let

$$R(\beta) = \{P_1, P_2, \dots, P_m\}$$

be an enumeration of $R(\beta)$ such that:

(i) the vectors P_i, $i = 1, 2, \dots, n$, are linearly independent and
(ii) the points P_i, $i = 1, 2, \dots, d$, are the vertices of K.

As an application of Theorems 3.6, 3.3, 3.7 and 3.10 we obtain the following:

Theorem 3.19 (The case of $\mathbb{R}^l$). *Suppose that $\Omega = \{1, 2, \dots, l\}$ and that the above assumptions are satisfied. Then*

(i) *X is a sublattice of $\mathbb{R}^l$ if and only if $R(\beta)$ contains exactly n points (i.e., $m = n$).*
(ii) *X is a lattice-subspace of $\mathbb{R}^l$ if and only if the polytope K has n vertices (i.e., $d = n$).*
(iii) *Let $m > n$. If $I_k = \beta^{-1}(P_k)$, and*

$$x_k = \sum_{i \in I_k} \|r(i)\|_1 \, e_i, \quad k = n+1, n+2, \dots, m,$$

then

$$Z = [x_1, \dots, x_n, x_{n+1}, \dots, x_m]$$

is the sublattice generated by $x_1, x_2, \dots, x_n$ and $\dim Z = m$.
(iv) *Let $d > n$. If $\xi_i : D(\beta) \to \mathbb{R}_+$, $i = 1, 2, \dots, d$, such that $\sum_{i=1}^{d} \xi_i(j) = 1$ and $\beta(j) = \sum_{i=1}^{d} \xi_i(j) P_i$ for each $j \in D(\beta)$, and x_{n+i}, $i = 1, 2, \dots, d-n$, are the following vectors of $\mathbb{R}^l$:*

$$x_{n+i} = \sum_{j \in D(\beta)} \xi_{n+i}(j) \, \|r(j)\|_1 \, e_j,$$

then

$$Y = [x_1, \dots, x_n, x_{n+1}, \dots, x_d]$$

is a minimal lattice-subspace of $\mathbb{R}^l$ containing $x_1, x_2, \dots, x_n$ and $\dim Y = d$.

In the following result Ω is again a compact, Hausdorff, topological space.

Theorem 3.20. *Let $K = \operatorname{co} \overline{R(\beta)}$ and let L be the set of finite-dimensional minimal lattice-subspaces of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$. Then the following statements are equivalent:*

(i) *K is a polytope with m vertices.*
(ii) *$L \neq \emptyset$ and $\dim Y = m$, for each $Y \in L$.*
(iii) *$L \neq \emptyset$.*

Proof. Suppose that (i) is true. Then by Theorem 3.10, there exists $Y \in L$ with $\dim Y = m$. Suppose that $Z \in L$ and $\{b_1, b_2, \dots, b_l\}$ is a positive basis of Z. Let

$$x_i = \sum_{j=1}^{l} \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_j = \sum_{i=1}^{n} \lambda_{ij}, \quad j = 1, 2, \dots, l,$$

$$\Phi = \{ j \mid \sigma_j \neq 0 \} \quad \text{and}$$

$$P_i = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi.$$

Then by Theorem 3.8 $P_i \in K$ for each $i \in \Phi$ and the vertices of K are among the points $P_i, i \in \Phi$; therefore there exist $i_1, i_2, \dots, i_m \in \Phi$ such that $P_{i1}, P_{i2}, \dots, P_{im}$

are the vertices of K. Also $n \leq m \leq l$. Let $T : Z \to \mathbb{R}^l$ such that $T(\sum_{i=1}^{l} \xi_i b_i) = \sum_{i=1}^{l} \xi_i e_i$ and let $y_i = T(x_i)$, $i = 1, 2, \dots, n$. The basic curve b of $y_1, y_2, \dots, y_n$ is:

$$b(i) = \frac{1}{\sigma_i}(\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

with range

$$R(b) = \{P_i \mid i \in \Phi\}.$$

So $R(b)$ is a subset of K containing the vertices of K; therefore

$$K = \operatorname{co} R(b).$$

Hence $\operatorname{co} R(b)$ is a polytope with vertices $P_{i1}, P_{i2}, \dots, P_{im}$. By the previous theorem, there exists an m-dimensional lattice-subspace F of $\mathbb{R}^l$ containing $y_1, y_2, \dots, y_n$. If $G = T^{-1}(F)$, then G is a lattice-subspace of Z and therefore also of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$. Since Z is minimal, we have that $G = Z$, and therefore $\dim Z = \dim F = m$. Hence we have shown that (i) $\Rightarrow$ (ii).

Suppose now that the statement (ii) is true. Let $Y \in L$ and $K = \operatorname{co} \overline{R(\beta)}$. Then by Theorem 3.8, K is a polytope with k vertices and

$$n \leq k \leq m.$$

By Theorem 3.10 there exists $Z \in L$ with $\dim Z = k$. By our assumption we have that $k = m$; therefore K has m vertices. Hence (ii) $\Rightarrow$ (i).

Also (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) by Theorem 3.8. □

In the following example we construct the sublattice Z generated by a four-dimensional subspace X of $\mathbb{R}^7$ as well as two minimal lattice-subspaces Y and Y' which contain X. It is remarkable that $Y \cap Y'$ is not a lattice-subspace as well as that both Y and Y' are not subspaces of Z.

Example 3.21. Let

$$\begin{aligned} x_1 &= (1,2,1,0,1,1,4), \\ x_2 &= (0,1,1,1,1,0,2), \\ x_3 &= (2,1,0,1,1,1,2), \\ x_4 &= (1,0,1,1,1,0,0), \end{aligned}$$

and let $X = [x_1, x_2, x_3, x_4]$. Let r be the curve and β the basic curve of x_i, $i = 1,2,3,4$. Then $r(1) = (1,0,2,1)$, $r(2) = (2,1,1,0)$, $r(3) = (1,1,0,1)$, $r(4) = (0,1,1,1)$, $r(5) = (1,1,1,1)$, $r(6) = (1,0,1,0)$, $r(7) = (4,2,2,0)$ and $\beta(1) = \frac{1}{4}(1,0,2,1)$, $\beta(2) = \beta(7) = \frac{1}{4}(2,1,1,0)$, $\beta(3) = \frac{1}{3}(1,1,0,1)$, $\beta(4) = \frac{1}{3}(0,1,1,1)$, $\beta(5) = \frac{1}{4}(1,1,1,1)$, $\beta(6) = \frac{1}{2}(1,0,1,0)$. In order to enumerate $R(\beta)$ as in Theorem 3.19 we remark the following:

(i) The vectors $P_1 = \beta(4)$, $P_2 = \beta(1)$, $P_3 = \beta(6)$ and $P_4 = \beta(3)$ are linearly independent.

(ii) Let $\beta(2) = P_5$. Then it is easy to show that for any proper subset Φ of $\{P_1, P_2, P_3, P_4, P_5\}$, $\operatorname{co}\Phi \neq \operatorname{co}\{P_1, P_2, P_3, P_4, P_5\} = K$; therefore P_i, $i = 1,2,3,4,5$, are vertices of the polytope K.

(iii) It is easy also to show that

$$\beta(5) = \frac{3(1-\theta)}{8} P_1 + \theta P_2 + \frac{1-5\theta}{4} P_3 + \frac{3(1-\theta)}{8} P_4 + \theta P_5. \tag{10}$$

Hence for any $\theta \in [0, \frac{1}{5}]$ the vector $P_6 = \beta(5)$ is a convex combination of $P_i,\ i = 1, 2, 3, 4, 5$; therefore $P_6 \in K$.

Hence

$$R(\beta) = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

and in accordance with the notations of Theorem 3.19, $n = 4$, $d = 5$ and $m = 6$. Since $n < d$, X is not a lattice-subspace and therefore also X is not a sublattice of $\mathbb{R}^7$. Let Z be the sublattice of $\mathbb{R}^7$ generated by x_1, x_2, x_3, x_4. In order to determine Z we define the sets

$$I_5 = \beta^{-1}(P_5) = \{2, 7\}, \quad I_6 = \beta^{-1}(P_6) = \{5\}$$

and the vectors

$$x_5 = \|r(2)\|_1\, e_2 + \|r(7)\|_1\, e_7 = 4e_2 + 8e_7$$

and

$$x_6 = \|r(5)\|_1\, e_5 = 4e_5.$$

Then by the theorem

$$Z = [x_1, x_2, x_3, x_4, x_5, x_6].$$

By Theorem 3.3 a positive basis $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ of Z is given by the formula

$$(b_1, b_2, b_3, b_4, b_5, b_6)^T = A^{-1}\,(x_1, x_2, x_3, x_4, x_5, x_6)^T,$$

where A is the 6×6 matrix with columns the vectors $\gamma(i),\ i = 1, 2, \dots, 6$, and γ is the basic curve of the vectors $x_i,\ i = 1, 2, \dots, 6$. So after the computations we find that $b_1 = 4e_1$, $b_2 = 8e_2 + 16e_7$, $b_3 = 3e_3$, $b_4 = 3e_4$, $b_5 = 8e_5$ and $b_6 = 2e_6$.

To determine a minimal lattice-subspace define the vectors $\xi_i,\ i = 1, 2, 3, 4, 5$, of $\mathbb{R}^7$ such that

$$\sum_{i=1}^{5} \xi_i(j) = 1 \quad \textit{and} \quad \beta(j) = \sum_{i=1}^{5} \xi_i(j) P_i, \qquad \text{for each } j = 1, 2, \dots, 7.$$

$$\begin{array}{lll}
\beta(1) = P_2 = \sum_{i=1}^5 \xi_i(1) P_i & \Rightarrow & \xi_2(1) = 1 \text{ and } \xi_k(1) = 0 \text{ for } k \neq 2. \\
\beta(2) = P_5 = \sum_{i=1}^5 \xi_i(2) P_i & \Rightarrow & \xi_5(2) = 1 \text{ and } \xi_k(2) = 0 \text{ for } k \neq 5. \\
\beta(3) = P_4 = \sum_{i=1}^5 \xi_i(3) P_i & \Rightarrow & \xi_4(3) = 1 \text{ and } \xi_k(3) = 0 \text{ for } k \neq 4. \\
\beta(4) = P_1 = \sum_{i=1}^5 \xi_i(4) P_i & \Rightarrow & \xi_1(4) = 1 \text{ and } \xi_k(4) = 0 \text{ for } k \neq 1. \\
\beta(5) = P_6 = \sum_{i=1}^5 \xi_i(5) P_i & \Rightarrow & \xi_1(5) = \xi_4(5) = \frac{3\,(1-\theta)}{8},\ \xi_2(5) = \xi_5(5) = \theta, \\
 & & \xi_3(5) = \frac{1-5\theta}{4}, \quad \text{by (10)}. \\
\beta(6) = P_3 = \sum_{i=1}^5 \xi_i(6) P_i & \Rightarrow & \xi_3(6) = 1 \text{ and } \xi_k(6) = 0 \text{ for } k \neq 3. \\
\beta(7) = P_2 = \sum_{i=1}^5 \xi_i(7) P_i & \Rightarrow & \xi_2(7) = 1 \text{ and } \xi_k(7) = 0 \text{ for } k \neq 2.
\end{array}$$

Define also the vector

$$\begin{aligned}
y_5 &= \sum_{j=1}^{7} \xi_5(j)\, \|r(j)\|_1\, e_j = \|r(2)\|_1\, e_2 + \theta \|r(5)\|_1\, e_5 \\
&= 4e_2 + 4\theta e_5, \quad \theta \in [0, 1/5].
\end{aligned}$$

Suppose that $\theta > 0$ in y_5 and that y_5' is the vector corresponding to $\theta = 0$, i.e., $y_5' = 4e_2$. Then the subspaces

$$Y = [x_1, x_2, x_3, x_4, y_5] \quad \textit{and} \quad Y' = [x_1, x_2, x_3, x_4, y_5']$$

are minimal lattice-subspaces containing the vectors x_i. Since the vectors $x_1, x_2, x_3, x_4, y_5, y_5'$ are linearly independent, we have $Y \neq Y'$. Also $X = Y \cap Y'$ is not a lattice-subspace. An important remark is that the vectors y_5, y_5' do not belong to Z. To show this suppose that $y_5 \in Z$. Then $y_5 \in Z_+$, and therefore

$$y_5 = \sum_{i=1}^{6} \lambda_i b_i, \quad \text{with } \lambda_i \in \mathbb{R}_+ \text{ for each } i.$$

This implies that $\lambda_2 = 1/2$ and $\lambda_2 = 0$, contradiction. Hence $y_5 \notin Z$. Also $y_5' \notin Z$. Therefore Y, Y' are not subspaces of Z.

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